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LETTER TO THE EDITOR

New integral solutions of the non-polynomial oscillator $V(x)=x^2+\lambda x^2/(1+gx^2)$ when $\lambda=2g(2-3g)$

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Abstract. We find exact solutions of the integral form and the corresponding eigenvalues for the non-polynomial oscillator $V(x)=x^2+\lambda x^2/(1+gx^2)$. The solutions are shown to exist when the parameter λ is constrained to satisfy a relation (of supersymmetric origin) involving the coupling constant g.

During the last few years the non-polynomial oscillator represented by the potential $V(x)=x^2+\lambda x^2/(1+gx^2)$ has been studied in considerable detail. In particular, exact analytical solutions have been derived by a number of authors (Lai and Lin 1982, Flessas 1981). A set of new solutions have also been found through the use of supersymmetric quantum mechanics (Roy and Roychoudhury 1987). However, all these solutions are of the elementary type and in the present letter we shall derive exact solutions in the form of integrals for the non-polynomial oscillator. The basic strategy that will be adopted in finding these integral solutions is to locate zero-energy solutions of the Schrödinger equation and to this end we find it convenient to use the formulism of supersymmetric quantum mechanism (SUSYQM). It may be pointed out that the present method can also be applied to a wide class of potentials and we have chosen the non-polynomial oscillator as an example because of its popularity and importance.

Let us recall that in one dimension a SUSYQM system is described by the following Hamiltonian (Witten 1981, Coopeı and Freedman 1983):

$$H=\begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{1}$$

$$H_\pm=-\mathrm{d}^2/\mathrm{d}x^2+V_\pm(x) \tag{2}$$

$$V_\pm(x)=W^2(x)\pm W'(x) \tag{3}$$

where $W(x)$ is called the superpotential. The eigenvalue equations corresponding to the pair of Hamiltonians given in (2) can be written as

$$H_\pm\varphi_\pm=E_\pm\varphi_\pm. \tag{4}$$

It can easily be shown that the zero-energy solutions of (4) are given by

$$\varphi_\pm^0\sim\exp\left(\pm\int^x W(t)\,\mathrm{d}t\right). \tag{5}$$

It may be that one of the $\varphi_\pm^0(x)$ is normalisable or that neither of them is normalisable. In order to find zero-energy solutions it is necessary to find the superpotential $W(x)$.

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The superpotential has to be chosen in such a way that one of the scalar potentials $V_\pm(x)$ could be identified with the non-polynomial oscillator. To this end we choose the following superpotential:

$$W(x) = x + \frac{2gx}{1+gx^2} + \frac{c}{x}. \tag{6}$$

From (3) the scalar potentials corresponding to (6) are found to be

$$V_-(x) = x^2 + \frac{8g^2x^2}{(1+gx^2)^2} + \frac{4cg-2g-4}{(1+gx^2)} + \frac{c(c+1)}{x^2} + (3+2c) \tag{7}$$

$$V_+(x) = x^2 + \frac{4cg+2g-4}{(1+gx^2)} + \frac{c(c-1)}{x^2} + (5+2c). \tag{8}$$

We intend to identify the Fermi sector, i.e. $V_+(x)$, with the non-polynomial oscillator represented by

$$V(x) = x^2 + \frac{\lambda x^2}{(1+gx^2)} = x^2 + \frac{\lambda}{g} - \frac{\lambda}{g(1+gx^2)}. \tag{9}$$

It follows that if (8) and (9) are identical then the following relations should hold:

$$-\lambda/g = 4cg + 2g - 4 \tag{10}$$

$$c = 0, 1 \tag{11}$$

and the relation between the energy eigenvalues is given by

$$E_{\mathrm{NP}} - \lambda/g = E_+ - (2c+5). \tag{12}$$

At this point we note that for $W(x)$ given by (8), the functions $\varphi^0_\pm(x)$ are given by

$$\varphi^0_-(x) \sim x^{-c}(1+gx^2)^{-1}\exp(-x^2/2) \tag{13}$$

$$\varphi^0_+(x) \sim x^{c}(1+gx^2)\exp(x^2/2). \tag{14}$$

From (13) and (14) it is seen that if $c = 1$ none of these functions belong to $L^2(-\infty, \infty)$. On the other hand if $c = 0$, $\varphi^0_-(x)$ satisfies proper boundary conditions at $x = \pm\infty$ and a physically acceptable ground state exists. SUSY is unbroken in this case. It is known that in the case of unbroken SUSY the ground state is unique (Roy and Roychoudhury 1986a) and therefore for $c = 0$ it is not possible to build up yet another physically acceptable ground state. We shall therefore stick to the case $c = 1$.

Let us now consider the following Riccati equation:

$$W_1^2(x) + W_1'(x) = W^2(x) + W'(x). \tag{15}$$

A trivial solution of this equation is

$$W_1(x) = W(x) \tag{16}$$

and a non-trivial solution is given by (Roy and Roychoudhury 1986b)

$$W_1(x) = W(x) + u(x) \tag{17}$$

where $u(x)$ is given by

$$u(x) = \exp\left(-2\int^x W(t)\,\mathrm{d}t\right)\left[K + \int^x \exp\left(-2\int^y W(t)\,\mathrm{d}t\right)\mathrm{d}y\right]^{-1} \tag{18}$$

K being a constant of integration. Therefore, a second zero-energy solution is given by

$$\Phi_+^0(x) \sim \exp\left(\int^x W_1(t)\,dt\right) = \exp\left(\int^x W(t) + u(t)\,dt\right) \tag{19}$$

and using the expression for $W(x)$ given in (6) we find

$$\Phi_+^0(x) \sim x(1+gx^2)\exp(x^2/2)\left(K + \int_{x_0}^x y^{-2}(1+gy)^{-2}\exp(-y^2)\,dy\right) \tag{20}$$

where x_0 is a suitable normalisation point.

However, before we admit (25) as a physically acceptable solution, it is necessary to show that $\Phi_+^0(x) \to 0$ as $x \to \pm\infty$. To do this we note that if K is suitably chosen, i.e. if

$$K = \int_{x_0^2}^\infty u^{-1/2}\left(\frac{2g}{(1+gu)^3} + \frac{1}{(1+gu)^2}\right)\exp(-u)\,du - x_0^{-1}(1+gx_0^2)^{-2}\exp(-x_0^2) \tag{21}$$

then we can write $\Phi_+^0(x)$ in the form

$$\Phi_+^0(x) \sim x(1+gx^2)\exp(x^2/2)\left[\int_{x^2}^\infty u^{-1/2}\left(\frac{2g}{(1+gu)^3} + \frac{1}{(1+gu)^2}\right)\exp(-u)\,du - x^{-1}(1+gx^2)^{-2}\exp(-x^2)\right]. \tag{22}$$

Now using the relation (Gradshteyn and Ryzhik 1980):

$$\Gamma(\alpha, x^2) = \int_{x^2}^\infty \exp(-t)t^{\alpha-1}\,dt \underset{x\to\pm\infty}{\sim} (x^2)^{\alpha-1}\exp(-x^2) \tag{23}$$

it can be easily shown that

$$\lim_{x\to\pm\infty} \Phi_+^0(x) = 0.$$

Furthermore, $\Phi_+^0(x)$ has no singularity at $x = 0$. Hence an exact integral solution of the non-polynomial oscillator is given by (22) and the corresponding energy eigenvalue is given by

$$E_{NP} = -(3+6g). \tag{24}$$

Finally, we show how more integral solutions to this problem can be obtained. First we note that the general form of the superpotential of the type (6) is (Roy and Roychoudhury 1987)

$$W(x) = x + \sum_{i=0}^{N} \frac{2g_i x}{(i+gx^2)} + \frac{1}{x}. \tag{25}$$

For $N = 0$ we obtain the solution (22). If we take $N = 1, 2, \ldots$ we can obtain more solutions, the procedure in each case being the same as that presented here.

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